Sections 13.1-13.2

Vector Functions and Space Curves

Vector Function: A vector-valued function $\mathbf{r}$ is a function whose domain is a set of real numbers and whose range is a set of vectors. We write $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.

Limits. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$ provided the limits of the component functions exist.
We say that $\mathbf{r}$ is continuous at $t = a$ if and only if its component functions are continuous at $t = a$.

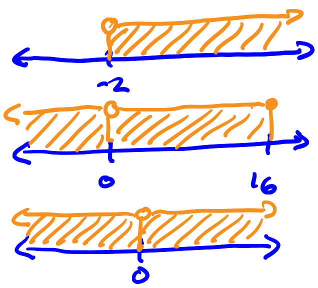
Ex1. If $\mathbf{r}(t) = \left\langle \ln(t+2), \frac{4-\sqrt{16-t}}{t}, \frac{\sin^4(t)}{t^4}\right\rangle$, find the domain of $\mathbf{r}$ and calculate $\lim_{t\to 0} \mathbf{r}(t)$.

Domain

•) $t+2>0 \Rightarrow t>-2$

•) $t\neq 0$ and $16-t\geq 0$
$(t\leq 16)$

•) $t\neq 0$



so, the domain of $\vec{r}$ is
$(-2,0)\cup(0,16]$

Limit

•) $\lim_{t\to 0} \ln(t+2) = \ln(2)$

•) $\lim_{t\to 0} \frac{4-\sqrt{16-t}}{t} \cdot \frac{4+\sqrt{16-t}}{4+\sqrt{16-t}} = \lim_{t\to 0} \frac{16-(16-t)}{t(4+\sqrt{16-t})} = \lim_{t\to 0} \frac{1}{4+\sqrt{16-t}} = \frac{1}{8}$

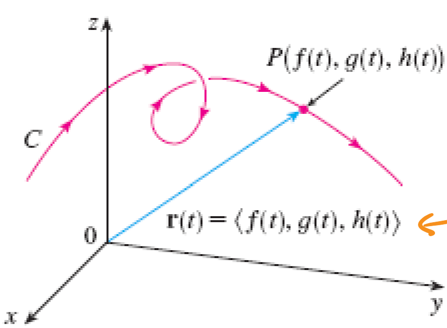
•) $\lim_{t\to 0} \frac{\sin^4 t}{t^4} = \lim_{t\to 0}\left(\frac{\sin t}{t}\right)^4 = (1)^4 = 1.$
$\hookrightarrow 1$

so, $\lim_{t\to 0} r(t) = \langle \ln(2), \frac{1}{8}, 1\rangle.$

Space Curves: Suppose f, g, and h are continuous functions on an interval I, then the set C of all *points* (x, y, z) in space, where $x = f(t), y = g(t), z = h(t)$ and t varies throughout the interval I is called space curve. The given equations are called parametric equations of C, and t is called a parameter.

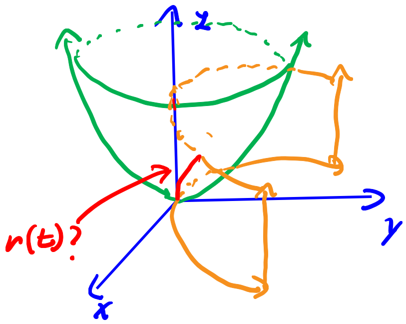
Parametric Equations of C

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$



$\leftarrow$ vector function $\vec{r}$ or vector equation or parametrization of the curve.

Ex2. Find a vector function, $\mathbf{r}(t)$, that represents the curve of intersection of the paraboloid $z = x^2 + 3y^2$ and the parabolic cylinder $y = x^2$.

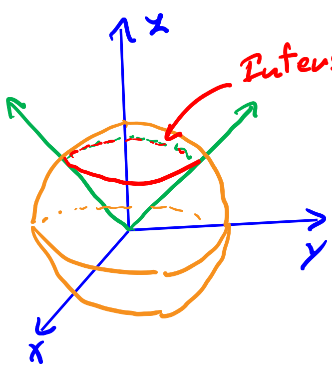


$$\begin{cases} x = t \\ y = t^2 \\ z = (t)^2 + 3(t^2)^2 = t^2 + 3t^4 \end{cases}$$

then

$$r(t) = \langle \underbrace{t}_{x(t)}, \underbrace{t^2}_{y(t)}, \underbrace{t^2 + 3t^4}_{z(t)} \rangle$$

Ex3. Let C be the curve of intersection of the cone $z = \sqrt{3x^2 + 3y^2}$ and the sphere $x^2+y^2+z^2 = 8$. Sketch and provide a parametrization of the curve C.



Intersection of $z = \sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2=8$

$\Rightarrow z^2 = 3x^2+3y^2$ and $x^2+y^2+z^2=8$

then $x^2+y^2+(3x^2+3y^2)=8$

$4x^2+4y^2=8$

$x^2+y^2=2$ $\leftarrow$ xy-plane: circle with center $(0,0)$ and radius $\sqrt{2}$.

$$\begin{cases} x = \sqrt{2}\cos(t) \\ y = \sqrt{2}\sin(t) \end{cases}$$

then $z = \sqrt{3(x^2+y^2)} = \sqrt{3(2)} = \sqrt{6}$.

so, $r(t) = \langle \sqrt{2}\cos(t), \sqrt{2}\sin(t), \sqrt{6} \rangle$.

Extra Note (check!). cone: $z = \sqrt{3x^2+3y^2} \Rightarrow \sqrt{6} \stackrel{?}{=} \sqrt{3(\sqrt{2}\cos t)^2 + (3(\sqrt{2}\sin t)^2}$

$= \sqrt{6(\cos^2 t + \sin^2 t)}$ ✓

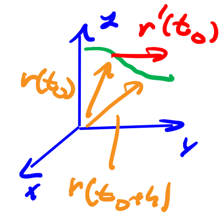
sphere: $x^2+y^2+z^2=8 \Rightarrow (\sqrt{2}\cos t)^2 + (\sqrt{2}\sin t)^2 + (\sqrt{6})^2 \stackrel{?}{=} 8$

$2\cos^2 t + 2\sin^2 t + 6 = 8$ ✓

Velocity and Tangent vectors

DEF. Let $\mathbf{r}$ be a vector function defined on $[a, b]$, the velocity (vector) of $\mathbf{r}$ at the time t_0 is given by

$$\mathbf{r}'(t_0) = \lim_{h \to 0} \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h}$$



The speed of $\mathbf{r}$ at t_0 is $||\mathbf{r}'(t_0)||$, the length of the velocity vector.

Remarks

- The velocity vector $\mathbf{r}'(t_0)$ is also called tangent vector if $\mathbf{r}'(t_0)$ is not the zero vector.
- If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$.
- If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.

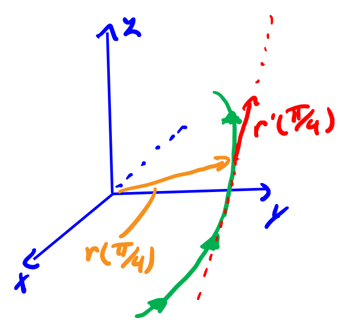
Ex4. The parametrization of a curve C is given by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ on the interval $[0, 4\pi]$. Find the velocity vector when $t = \pi/4$.

Goal: $r'(\pi/4)$

First, $r'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

then, $r'(\pi/4) = \langle -\sin(\frac{\pi}{4}), \cos(\frac{\pi}{4}), 1 \rangle$

$= \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \rangle$



- Also, find parametric equations for the tangent line to the curve C at the point where $t = \pi/4$.

•) when $t = \pi/4$, $r(\pi/4) = \langle \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}), \frac{\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$

the point at $\pi/4$ is $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4})$.

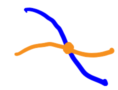
•) a vector parallel to the tangent line at $\pi/4$ is $r'(\frac{\pi}{4})$.

$r'(\pi/4) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 \rangle$.

•) Parametric eq. of the tangent line at $t = \pi/4$ are:

$$x = \frac{\sqrt{2}}{2} + s\left(-\frac{\sqrt{2}}{2}\right)$$
$$y = \frac{\sqrt{2}}{2} + s\left(\frac{\sqrt{2}}{2}\right)$$
$$z = \frac{\pi}{4} + s\,(1)$$

where s is a parameter (defined by the definition of parametric equations of a line)



Angle of intersection between two curves.
To find the angle of intersection, first find the intersection point of the curves.
If the curves intersect, the angle of intersection is the angle between their tangent vectors at that point.

Ex5. At what point do the curves $\mathbf{r}_1(t) = \langle 2t, 1+t, 3+t^2\rangle$ and $\mathbf{r}_2(s) = \langle 1+s, s, s^2-2\rangle$ intersect? Find the cosine of their angle of intersection.

•) we want
$$\begin{cases} 2t = 1+s & (i) \\ 1+t = s & (ii) \\ 3+t^2 = s^2-2 & (iii) \end{cases}$$

From (i) and (ii): $2t = 1 + (1+t) \Rightarrow \boxed{t=2}$

in (ii): $s = 1+t = 1+2 = 3 \Rightarrow \boxed{s=3}$

Check (iii): $3 + (2)^2 = (3)^2 - 2$?

$7 = 7$ ✓

Then the intersection point is $(4, 3, 7)$.

•) then, we need $r_1'(t) = \langle 2, 1, 2t\rangle$, $r_2'(s) = \langle 1, 1, 2s\rangle$

$r_1'(2) = \langle 2, 1, 4\rangle$

$r_2'(3) = \langle 1, 1, 6\rangle$

so, $\cos(\theta) = \dfrac{r_1'(2) \cdot r_2'(3)}{\|r_1'(2)\| \cdot \|r_2'(3)\|} = \dfrac{2+1+24}{\sqrt{21} \cdot \sqrt{38}} = \dfrac{27}{\sqrt{21} \cdot \sqrt{38}}$

Extra Question: Do the particles collide?

we have $r_1(t) = \langle 2t, 1+t, 3+t^2\rangle$, $r_2(t) = \langle 1+t, t, t^2-2\rangle$.

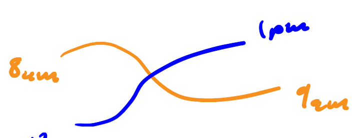
We want
$$\begin{cases} 2t = 1+t & (i) \\ 1+t = t & (ii) \\ 3+t^2 = t^2-2 & (iii) \end{cases}$$

From (i): $2t - t = 1 \Rightarrow t = 1$

then in (ii) $1+1 = 1 \Rightarrow 2 = 1$ "Impossible"

in (iii) $3+1 = 1-2$ "Impossible"

particles do NOT collide.



- they don't collide bc of the times

Derivatives and Integrals of Vector Functions

Given a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, we define the following.

- Velocity vector: $v(t) := \mathbf{r}'(t)$
- $\displaystyle\int_a^b \mathbf{r}(t)dt := \Big(\int_a^b x(t)\ dt\Big)\mathbf{i} + \Big(\int_a^b y(t)\ dt\Big)\mathbf{j} + \Big(\int_a^b z(t)\ dt\Big)\mathbf{k}$

Differentiation Rules Let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ be differentiable vector functions in $\mathbb{R}^3$ and $p(t)$ be a differentiable scalar function:

$p: \mathbb{R} \to \mathbb{R}$

Ex. $p(t) = \underbrace{t^2+2}_{\text{Output is a real number}}$

Input is a real number

- Sum Rule: $\dfrac{d}{dt}[\mathbf{a}(t) + \mathbf{b}(t)] = \mathbf{a}'(t) + \mathbf{b}'(t)$
- Scalar Multiplication: $\dfrac{d}{dt}[p(t)\mathbf{a}(t)] = p'(t)\mathbf{a}(t) + p(t)\mathbf{a}'(t)$
- Dot Product: $\dfrac{d}{dt}[\mathbf{a}(t)\cdot\mathbf{b}(t)] = \mathbf{a}'(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\mathbf{b}'(t)$
- Cross Product: $\dfrac{d}{dt}[\mathbf{a}(t)\times\mathbf{b}(t)] = \mathbf{a}'(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\mathbf{b}'(t)$
- Chain Rule: $\dfrac{d}{dt}[\mathbf{a}\big(p(t)\big)] = p'(t)\mathbf{a}'\big(p(t)\big)$

Dot product

Ex6. Suppose $f(t) = \mathbf{u}(t)\cdot\mathbf{v}(t)$, $\mathbf{u}(2) = \langle 1, 2, -1\rangle, \mathbf{u}'(2) = \langle 3, 0, 4\rangle$, and $\mathbf{v}(t) = \langle t, t^2, t^3\rangle$. Find $\underline{f'(2)}$.

$v'(t) = \langle 1, 2t, 3t^2\rangle$

$v'(2) = \langle 1, 4, 12\rangle$

$f'(t) = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

$f'(2) = u'(2)\cdot v(2) + u(2)\cdot v'(2)$

$f'(2) = \underbrace{\langle 3,0,4\rangle \cdot \langle 2,4,8\rangle}_{} + \underbrace{\langle 1,2,-1\rangle\cdot\langle 1,4,12\rangle}_{}$

$= (6+0+32) \quad + \quad (1+8-12) = \boxed{35}$

Ex7. If $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2t)\mathbf{k}$, compute $\displaystyle\int_0^{\pi/2} \mathbf{r}(t)\ dt$.

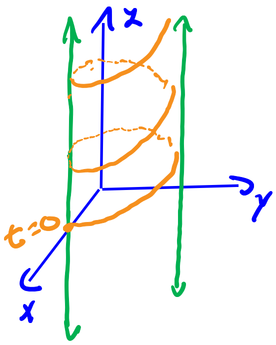
$$\int_0^{\pi/2} r(t)\,dt = \left\langle \int_0^{\pi/2} 2\cos(t)\,dt,\ \int_0^{\pi/2} \sin(t)\,dt,\ \int_0^{\pi/2} 2t\,dt \right\rangle$$

$$= \left\langle 2\sin\Big|_0^{\pi/2},\ -\cos\Big|_0^{\pi/2},\ t^2\Big|_0^{\pi/2} \right\rangle$$

$$= \left\langle 2-0,\ -0-(-1),\ \frac{\pi}{4}^2 - 0 \right\rangle = \left\langle 2, 1, \frac{\pi}{4}^2 \right\rangle.$$

Exercises

1. Sketch the curve $\mathbf{r}(t) = \big\langle \cos(t), \sin(t), t \big\rangle$ defined on the interval $[0, 4\pi]$.



$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$ "Helix"

$t=0: \quad r(0) = \langle 1, 0, 0 \rangle$

$t=\pi/2: \quad r(\pi/2) = \langle 0, 1, \pi/2 \rangle$

$t=\pi: \quad r(\pi) = \langle -1, 0, \pi \rangle$

$t=3\pi/2: \quad r(3\pi/2) = \langle 0, -1, 3\pi/2 \rangle$

$t=2\pi: \quad r(2\pi) = \langle 1, 0, 2\pi \rangle$

2. Let $\mathbf{a}(t)$ be a differentiable vector function such that $\mathbf{a}(t) \neq \vec{0}$ for all t. Prove the following identities:

i) $\displaystyle\frac{d}{dt}\Big\{||\mathbf{a}(t)||\Big\} = \frac{\mathbf{a}(t)\cdot\mathbf{a}'(t)}{||\mathbf{a}(t)||}$ ii) $\displaystyle\frac{d}{dt}\Big\{\frac{1}{||\mathbf{a}(t)||}\Big\} = -\frac{\mathbf{a}(t)\cdot\mathbf{a}'(t)}{||\mathbf{a}(t)||^3}$

$$\frac{d}{dt}\left\{(a(t)\cdot a(t))^{1/2}\right\} = \frac{1}{2}(a(t)\cdot a(t))^{-1/2}\frac{d}{dt}\{a(t)\cdot a(t)\}$$

$$= \frac{1}{2\sqrt{a(t)\cdot a(t)}}\left\{a'(t)\cdot a(t) + a(t)\cdot a'(t)\right\}$$

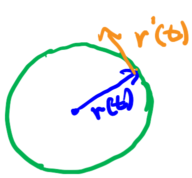
$$= \frac{1}{\cancel{2}||a(t)||}\left\{\cancel{2}\,a(t)\cdot a'(t)\right\} \checkmark$$

$$\frac{d}{dt}\left\{||a(t)||^{-1}\right\} = (-1)||a(t)||^{-2}\frac{d}{dt}\{||a(t)||\}$$

$$= \frac{-1}{||a(t)||^2}\left\{\frac{a(t)\cdot a'(t)}{||a(t)||}\right\}$$

$$= \frac{-a(t)\cdot a'(t)}{||a(t)||^3} \checkmark$$

3. If the scalar function $||\mathbf{r}(t)||$ is constant; i.e., $||\mathbf{r}(t)|| = c$ regardless the value of t, show that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.



If $c = 0 \Rightarrow ||r(t)|| = 0 \Rightarrow r(t) = \vec{0}$

If $c \neq 0 \Rightarrow \frac{d}{dt}||r(t)|| = \frac{d}{dt}c$

$$\frac{r(t)\cdot r'(t)}{||r(t)||} = 0 \Rightarrow r(t)\cdot r'(t) = 0$$

therefore $r(t) \perp r'(t)$.